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# Parametric representation of open quantum systems and crossover from quantum to classical environment

D. Calvani,<sup>1,2</sup> A. Cuccoli,<sup>1,2</sup> N.I. Gidopoulos,<sup>3</sup> and P. Verrucchi<sup>4,1,2</sup>

<sup>1</sup>*Dipartimento di Fisica, Università di Firenze, Via G. Sansone 1, I-50019 Sesto Fiorentino (FI), Italy*

<sup>2</sup>*INFN Sezione di Firenze, via G. Sansone 1, I-50019 Sesto Fiorentino (FI), Italy*

<sup>3</sup>*ISIS, STFC, Rutherford Appleton Laboratory, Didcot, OX11 0QX, United Kingdom*

<sup>4</sup>*Istituto dei Sistemi Complessi, Consiglio Nazionale delle Ricerche,  
via G. Sansone 1, I-50019 Sesto Fiorentino (FI), Italy*

The behaviour of most physical systems is affected by their natural surroundings. A quantum system with an environment is referred to as "open", and its study varies according to the classical or quantum description adopted for the environment. We propose an approach to open quantum systems that allows us to follow the crossover from quantum to classical environments; to achieve this, we devise an exact parametric representation of the principal system, based on generalized coherent states for the environment. The method is applied to the  $s=\frac{1}{2}$  Heisenberg-star with frustration, where the quantum character of the environment varies with the couplings entering the Hamiltonian  $H$ . We find that when the star is in an eigenstate of  $H$ , the central spin behaves as if it were in an effective magnetic field, pointing in the direction set by the environmental coherent-state angle variables  $(\theta, \varphi)$ , and broadened according to their quantum probability distribution. Such distribution is independent on  $\varphi$  while, as a function of  $\theta$ , is seen to get narrower as the quantum character of the environment is reduced, collapsing into a Dirac- $\delta$  function in the classical limit. In such limit, as  $\varphi$  is left undetermined, the Von Neumann entropy of the central spin remains finite. It is equal to the entanglement of the original fully-quantum model, which further establishes a relation between this latter quantity and the Berry-phase characterizing the dynamics of the central spin in the effective magnetic field.

## I. INTRODUCTION

Quantum systems with an environment are usually referred to as *open quantum systems* (OQS). Despite having been extensively studied since the very birth of quantum mechanics, the relevance acquired by certain features (such as coherence and entanglement) in the last decades, have boosted further the interest towards their behaviour, both in the dissipative and in the non-dissipative case<sup>1–4</sup>. The description of an OQS in terms of the corresponding reduced density matrix, defined as the partial trace of the density matrix of the global system over the Hilbert space of the environment, is axiomatically exact and fully retains the quantum character of the environment, making it the preferred approach, particularly in the realm of quantum information theory and computation<sup>5–8</sup>. However, there exists another description of OQS, where the principal system is in a pure state, but under the effect of a local Hamiltonian depending on "external" parameters, whose presence testifies the existence of a surrounding environment. At the heart of this approach stands the approximation that the environment be classical, so that the operators acting on its Hilbert space are replaced by c-number parameters; in this way, the interaction Hamiltonian is reduced to an effectively local one for the sole principal system. The prototypical example of this theoretical scheme is that of a spin in an external magnetic field, but many other examples can be easily spotted.

In the two approaches sketched above, the environment is either quantum or classical. Aim of this work is to devise a method for studying OQS in the non-

dissipative case, capable of interpolating between the quantum description of the environment and its classical limits, without affecting the quantum nature of the principal system. Inspired by the Born-Oppenheimer approach<sup>9</sup>, where electrons are described in terms of pure states parametrically dependent on the nuclear positions, we construct a representation of OQS in terms of pure states that depend on continuous parameters characterising the environment. Such representation is formally exact, in the same way that the parametric separation between electronic and nuclear subsystems can be made exact<sup>10,11</sup>, with the role of principal system and environment essentially interchangeable. Once the procedure for constructing the above parametric representation, which is based on the use of generalized coherent states for the environment, is devised, we implement it in the case of a specific spin model, namely the spin- $\frac{1}{2}$  star with frustration<sup>12,13</sup>: this system belongs to the large family of the so-called "central-spin" models, which have been extensively studied<sup>14–24</sup>, since they describe magnetic interactions that play a relevant role in the physics of candidate future nanodevices<sup>25–32</sup>.

The parametric representation, besides providing an insight into the physical behaviour of the above spin model, prove suitable for dealing with peculiar geometrical effects such as the Berry's phase<sup>33,34</sup>, and allows us to confront a very relevant topic pertaining OQS, namely the subtle connection between geometric aspects of quantum mechanics and entanglement<sup>35,36</sup>, as discussed in the final part of the paper.

## II. THE PARAMETRIC REPRESENTATION WITH COHERENT STATES

Let us consider an isolated system in a pure, normalized, state  $|\Psi\rangle$ : the system is made of a principal system  $A$  and its environment  $B$ , with separable Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. Given two orthonormal bases  $\{|\alpha\rangle\} \subset \mathcal{H}_A$  and  $\{|\beta\rangle\} \subset \mathcal{H}_B$ , it is  $|\Psi\rangle = \sum_{\alpha\beta} c_{\alpha\beta} |\alpha\rangle |\beta\rangle$ . Aimed at the interpolation scheme mentioned in the Introduction, we understand that describing the environment in terms of continuous parameters, rather than with a set of discrete ones, can be rewarding. The idea of introducing proper “environmental coherent states” naturally follows, and we accomplish it by resorting to the group-theoretical construction proposed by Gilmore (see Ref.<sup>37</sup> and references therein). Let the total environmental Hamiltonian be  $H_B + H_{AB}$ , where the first, local, term contains operators acting on  $\mathcal{H}_B$ , while the second, interaction, term contains operators acting on  $\mathcal{H}_A \otimes \mathcal{H}_B$ . We assume  $H_{AB}$  is a linear combination of tensor products of operators acting on  $\mathcal{H}_A$  and on  $\mathcal{H}_B$ , as is the case in most physical situations. Let  $G$  be the dynamical group in terms of whose generators we can write the environmental Hamiltonian; the Hilbert space  $\mathcal{H}_B$  is given by the specific physical set up and is associated with a unitary irreducible representation of  $G$ . The choice of a reference state  $|\beta_0\rangle \in \mathcal{H}_B$  fixes: *i*) the maximum stability subgroup  $F \subset G$ , i.e. the set of those operators  $f$  such that  $f|\beta_0\rangle = e^{i\lambda_f} |\beta_0\rangle$ ; *ii*) the quotient  $G/F$ , such that any  $g \in G$  can be locally decomposed as  $g = \Omega f$ , with  $f \in F$  and  $\Omega \in G/F$ . Generalized *coherent states*  $|\Omega\rangle$  are eventually defined as  $|\Omega\rangle = \Omega |\beta_0\rangle$ , with  $\Omega \in G/F$ ; they can be normalized, and form an overcomplete set in  $\mathcal{H}_B$ , *i.e.*

$$\int d\mu(\Omega) |\Omega\rangle \langle \Omega| = \mathbf{1}_{\mathcal{H}_B}, \quad (1)$$

with  $d\mu(\Omega)$  a proper group-invariant measure in  $G/F$ , and  $\langle \Omega | \Omega' \rangle \neq \delta(\Omega - \Omega')$ . Using the above resolution of the identity in  $\mathcal{H}_B$ , the state of the total system can be written as

$$|\Psi\rangle = \int d\mu(\Omega) \chi^\Psi(\Omega) |\Omega\rangle |\phi_A^\Psi(\Omega)\rangle, \quad (2)$$

with

$$|\phi_A^\Psi(\Omega)\rangle \equiv \frac{1}{\chi^\Psi(\Omega)} \sum_{\alpha} f_{\alpha}(\Omega) |\alpha\rangle, \quad (3)$$

$$f_{\alpha}(\Omega) \equiv \sum_{\beta} \langle \Omega | \beta \rangle c_{\alpha\beta}, \quad (4)$$

$$\chi^\Psi(\Omega) \equiv e^{i\Lambda(\Omega)} \sqrt{\sum_{\alpha} |f_{\alpha}(\Omega)|^2}, \quad (5)$$

where  $e^{i\Lambda(\Omega)}$  is a gauge freedom. Each ket  $|\phi_A^\Psi(\Omega)\rangle$  is a normalized element of  $\mathcal{H}_A$  and therefore represents a physical state for the principal system.

By parametric representation of  $A$  we will hereafter mean its description in terms of the pure states  $\{|\phi_A^\Psi(\Omega)\rangle\}$ : the dependence of  $|\phi_A^\Psi(\Omega)\rangle$  on the parameter  $\Omega$  is the fingerprint that an environment exists. Actually, it can be easily shown that  $|\phi_A^\Psi\rangle$  depends on the parameter  $\Omega$  if and only if the global state  $|\Psi\rangle$  is entangled: it is the entangled structure of  $|\Psi\rangle$  that causes the dependence on  $\Omega$  to be conveyed from the environment to the principal system, a fact that will play a crucial role in relating entanglement and geometrical phases, as discussed in the last Section.

The normalization of  $|\Psi\rangle$  implies  $\int d\mu(\Omega) |\chi^\Psi(\Omega)|^2 = 1$ , allowing  $|\chi^\Psi(\Omega)|^2$  to be interpreted as a probability distribution on  $G/F$ , which turns out to be the phase space of the environment under rather general assumptions<sup>37</sup>; indeed, by a simple calculation it is immediate to show that  $|\chi^\Psi(\Omega)|^2$  is just the Husimi  $Q$ -function of the environmental reduced density matrix<sup>37,38</sup>,  $\langle \Omega | (\text{Tr}_A |\Psi\rangle \langle \Psi|) | \Omega \rangle = |\chi^\Psi(\Omega)|^2$ .

The resulting picture for the principal system is that of a continuous collection of pure, parametrized, states whose occurrence is ruled by a probability distribution over the phase space of the environment. The relation  $\text{Tr}_B[\cdot] = \int d\mu(\Omega) \langle \Omega | \cdot | \Omega \rangle$  holds<sup>38</sup>, yielding, for the reduced density matrix of the principal system,  $\rho_A = \int d\mu(\Omega) |\chi(\Omega)|^2 |\phi_A^\Psi(\Omega)\rangle \langle \phi_A^\Psi(\Omega)|$ , and hence

$$\langle O \rangle \equiv \text{Tr}_A(\rho_A O) = \int d\mu(\Omega) |\chi(\Omega)|^2 \langle \phi_A^\Psi(\Omega) | O | \phi_A^\Psi(\Omega) \rangle, \quad (6)$$

for any principal-system observable  $O$ . The above construction provides an exact description for both the principal system and its environment, and further allows us to treat them in a very different formal scheme: in fact, and at variance with other related works<sup>17,39,40</sup>, coherent states are here exclusively adopted for the environment. This makes the approximations which naturally arise in the coherent state formalism available for describing the environment, but it also prevents the principal system from being affected by those same approximations.

Before ending this section, we notice that any resolution of the identity in the Hilbert space of the environment defines a parametric representation for the principal system. In fact, the construction of the parametric representation described above can be both generalized, referring to approaches to quantum mechanics on phase space<sup>41</sup> that go beyond the theory of coherent states, and made more specific, whenever continuous variables different from those related to coherent states emerge in dealing with specific types of environment<sup>42,43</sup>.

## III. THE SPIN- $\frac{1}{2}$ STAR WITH FRUSTRATION.

We apply the above formalism to the specific physical situation where a spin  $1/2$  ( $\sigma/2$ , hereafter called *qubit*) interacts with an even set of  $N$  spins  $1/2$  ( $s_i$ , hereafter called *environmental spins*) via an isotropic antiferro-

magnetic coupling<sup>12,13</sup>. The environmental spins interact among themselves and the total Hamiltonian is that of the so-called “spin- $\frac{1}{2}$  star with frustration”,

$$H = H_B + H_{AB}, \quad (7)$$

$$H_B = \frac{k}{N} \sum_i \mathbf{s}_i \cdot \mathbf{s}_{i+1}; \quad k > 0 \quad (8)$$

$$H_{AB} = \frac{g}{N} \frac{\boldsymbol{\sigma}}{2} \cdot \sum_i \mathbf{s}_i; \quad g > 0 \quad (9)$$

where  $i$  runs over the sites of the external ring. As mentioned in the introduction, the spin- $\frac{1}{2}$  star with frustration belongs to a class of “central-spin”-like models that have been extensively studied in the last decade. Without entering a detailed case study, which goes beyond the scope of this paper, we underline that this model possesses the most welcome property of allowing the quantum character of the environment, as measured by the total spin of the ring, to be varied acting on the ratio  $k/g$  (usually referred to as the *frustration ratio*), as discussed below. Moreover the model is exactly solvable and analytical expressions of its eigenstates and eigenvalues are available<sup>12</sup>.

Let us briefly review the main known facts about the spin- $\frac{1}{2}$  star with frustration, Hamiltonian (7)-(9): the integrals of motion are the total Hamiltonian  $H$ , the local environmental Hamiltonian  $H_B$ , the square  $\mathbf{J}^2$  of the total angular momentum  $\mathbf{J} \equiv \boldsymbol{\sigma}/2 + \sum_i \mathbf{s}_i$ , its component along the quantization axis,  $J^z$ , and the square  $\mathbf{S}^2$  of the total spin of the ring  $\mathbf{S} = \sum_i \mathbf{s}_i$ , with the respective eigenvalues  $E$ ,  $kE_B$ ,  $J(J+1)$ ,  $M$ , and  $S(S+1)$ ; it is also useful to define  $m$  as the eigenvalues of the component  $S^z$  of the total environmental spin  $\mathbf{S}$ . The relations  $J = S \pm \frac{1}{2}$  and  $M = m \pm \frac{1}{2}$  hold. The total spin  $S$  ranges from 0 to  $N/2$ . Apart from the state with  $S = 0$ , which has energy  $kE_B$ , for each assigned value  $S > 0$  the energy spectrum consists of the two multiplets

$$E_+ = kE_B + \frac{g}{2N} S, \quad (10)$$

$$E_- = kE_B - \frac{g}{2N} (S+1); \quad (11)$$

the lowest eigenvalue of  $H_B$  obeys the Lieb-Mattis ordering<sup>44</sup>,  $E_B(S) < E_B(S+1)$ , and the competition between the two terms in Eq. (11), embodying the antiferromagnetic frustration of the model, makes the ground-state (GS) of the star belong to the  $E_-$  multiplet with a value of  $S$  that varies with the frustration ratio  $k/g$ . In fact, for  $0 \leq k/g \leq 1/4 \equiv \alpha_0$ , the GS has  $S = N/2$  while, when  $k/g$  increases, there exists a sequence of critical values  $\alpha_n$ ,  $n = 1, \dots, (N/2 - 1)$ , such that  $S = N/2 - n$  for  $\alpha_{n-1} < k/g \leq \alpha_n$ , and  $S = 0$  for  $k/g > \alpha_{N/2-1} \gg 1$ . Notice that the critical values  $\alpha_n$  depend on  $N$ , with the exception of  $\alpha_0$  which equals  $1/4$  for all  $N$ . The basic structure of the eigenstates belonging to a subspace with fixed  $S$  is

$$|\Psi_M^\pm\rangle = a_M^\pm |\uparrow\rangle |m-\rangle + b_M^\pm |\downarrow\rangle |m+\rangle, \quad (12)$$

where  $|\uparrow, \downarrow\rangle$  are the eigenstates of  $\sigma^z$ ,  $|m_\pm\rangle$  are the eigenstates of  $S^z$  with eigenvalues  $m = M \pm 1/2$ , and the apex  $\pm$  refers to the state having energy  $E_\pm$ . Introducing  $\tilde{S} \equiv S + 1/2$ , the coefficients are

$$a_M^\pm = \pm \frac{1}{\sqrt{2}} \sqrt{1 \pm \frac{M}{\tilde{S}}}, \quad b_M^\pm = \frac{1}{\sqrt{2}} \sqrt{1 \mp \frac{M}{\tilde{S}}}, \quad (13)$$

yielding  $a^\pm = \pm b^\mp$ . The entanglement between the qubit and its environment for the states (12), as measured by the Von Neumann entropy, is easily found to be the same for both multiplets and to depend just on the ratio  $M/\tilde{S} \equiv \cos \vartheta_M$ , according to

$$\mathcal{E}_{\sigma S} = -h \left[ \frac{1}{2} (1 - \cos \vartheta_M) \right], \quad (14)$$

where  $h[x]$  ( $0 \leq x \leq 1$ ) is the binary entropy  $x \log x + (1-x) \log(1-x)$ .

We construct the parametric representation of the states (12), each having  $S$  fixed, by identifying  $\mathbf{S} = \sum_i \mathbf{s}_i$  with the environment and  $\boldsymbol{\sigma}$  with the principal system; the dynamical group  $G$  to be employed in the generalized coherent state construction is therefore  $SU(2)$ , with  $\mathcal{H}_B$  its spin- $S$  irreducible representation. We choose the reference state  $|\beta_0\rangle$  to be the maximal weight of the spin  $S$ -representation, *i.e.*  $|\beta_0\rangle = |m = S\rangle \in \mathcal{H}_B$ , so that  $U(1)$  is the maximum stability subgroup  $F$ , eventually ending up in the *Bloch coherent states*  $|\Omega\rangle$ , whose expansion over the  $S_z$  basis  $\{|m\rangle\} \in \mathcal{H}_B$  is (using the conventions of<sup>38</sup>)

$$|\Omega\rangle \equiv e^{\zeta S^- - \zeta^* S^+} |m = S\rangle = \sum_{m=-S}^S g_m(\theta) e^{i(S-m)\varphi} |m\rangle, \quad (15)$$

where  $S^\pm = S_x \pm iS_y$ ,  $\zeta = \frac{\theta}{2} e^{i\varphi}$ ,

$$g_m(\theta) \equiv \sqrt{\binom{2S}{S+m}} \cos^{S+m} \left( \frac{\theta}{2} \right) \sin^{S-m} \left( \frac{\theta}{2} \right), \quad (16)$$

and  $\Omega \equiv (\theta, \varphi)$  is a point of the  $S^2 \simeq SU(2)/U(1)$  sphere. The resolution of the identity reads  $\mathbf{1}_{\mathcal{H}_B} = \frac{\tilde{S}}{2\pi} \int d\Omega |\Omega\rangle \langle \Omega|$ , where  $d\Omega \equiv \sin \theta d\theta d\varphi$  is the euclidean measure on the  $S^2$  sphere. From the definitions (2)-(5) and the expansion (15), we can explicitly write the states (12) as

$$|\Psi_M^\pm\rangle = \frac{\tilde{S}}{2\pi} \int d\Omega \chi_M^\pm(\Omega) |\Omega\rangle |\phi_M^\pm(\Omega)\rangle, \quad (17)$$

where

$$\chi_M^\pm(\Omega) = e^{i(\tilde{S}-M)\varphi} \sqrt{[a_M^\pm g_{M-\frac{1}{2}}(\theta)]^2 + [b_M^\pm g_{M+\frac{1}{2}}(\theta)]^2}, \quad (18)$$

and the qubit parametrized states are

$$|\phi_M^\pm(\Omega)\rangle = \pm \cos \left( \frac{\Theta_M^\pm(\theta)}{2} \right) |\uparrow\rangle + \sin \left( \frac{\Theta_M^\pm(\theta)}{2} \right) e^{i\varphi} |\downarrow\rangle, \quad (19)$$

where  $\Theta_M^\pm(\theta)$  are the solution with respect to  $\Theta$  of the equations

$$\tan \frac{\Theta}{2} = \left( \tan \frac{\vartheta_M}{2} \right)^{\pm 1} \tan \frac{\vartheta_M}{2} \cot \frac{\theta}{2}. \quad (20)$$

Notice that although  $|\Omega\rangle$  depends on  $\varphi$  according to Eq.(15), the distribution  $|\chi_M^\pm(\Omega)|^2$  does not. This follows from the structure of the eigenstates (12), implying that only one term enters the sum in Eq.(5) and no interference effect consequently emerges in  $|\chi_M^\pm(\Omega)|^2$ . Such feature is specifically due to the symmetry of the Heisenberg interaction and plays an essential role when the classical limit of the ring is taken, as thoroughly discussed in the next Section.

The parametric representation of the qubit is given by the kets (19), each corresponding to the physical state the qubit is in, provided the total system is in  $|\Psi_M^\pm\rangle$  and the environment in the coherent state  $|\Omega\rangle$ . As  $|\chi_M^\pm(\Omega)|^2$  does not depend on  $\varphi$ , the  $\Omega$ -occurrence is ruled by the  $\theta$ -normalized probability distribution  $\tilde{S} \sin(\theta) |\chi_M^\pm(\Omega)|^2 \equiv p_M^\pm(\theta)$ , which can be shown to be the properly normalized  $Q$ -representation of the environmental density matrix. In the upper panel of Fig.1 we show  $p_M^-(\theta)$  for

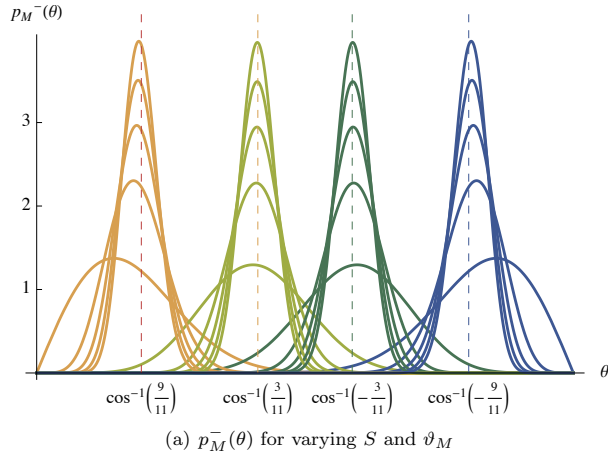


FIG. 1: Upper panel, (a):  $p_M^-(\theta)$  for  $\frac{M}{S} = \frac{9}{11}, \frac{3}{11}, -\frac{3}{11}, -\frac{9}{11}$  (from left to right) and  $S=5,16,27,38,49$  (from below); the vertical dashed lines mark the corresponding values of  $\vartheta_M$  (see text), each identified by a given color. Lower panel, (b-e): qubit-states distributions,  $\pi_M^-(\pi - \theta, \varphi)$ , on the Bloch sphere for  $S = 5$  and  $\vartheta_M$  as in the upper panel. A black line of latitude marks  $\vartheta_M$  on each sphere, and the corresponding value of  $M$  is reported below.

different values of  $M/\tilde{S}$ ; the values of  $S$  are chosen so as to consistently correspond to half-integer  $M$ . Two

effects clearly testify that we are dealing with a quantum environment: the finite width of the distribution and the position shift of its maximum with respect to  $\vartheta_M$ , such shift being the signature that the qubit exists. When the quantum character of the environment is reduced, *i.e.*  $S$  is increased with  $M/\tilde{S}$  fixed, both quantities lessen. The probability distribution of the qubit states (19) on the Bloch sphere, parametrized by  $\Theta$  and  $\varphi$ , is the integrand of Eq.(6) with  $O = |\phi_M^\pm\rangle\langle\phi_M^\pm|$ , and reads  $\pi_M^\pm(\Theta, \varphi) = p_M^\pm(\theta)$ , with  $\Theta$  and  $\theta$  connected by Eq.(20). When the total system is in its GS multiplet, it is easily seen that  $\Theta = \pi - \theta$  for all values of  $M$ . The lower panel of Fig.1 shows  $\pi_M^-(\Theta, \varphi) = p_M^-(\pi - \theta)$  for  $M/\tilde{S}$  as in the upper panel, and  $S = 5$  (each sphere is below the corresponding distribution); the angle  $\vartheta_M$  is identified by the black line of latitude. The conditional

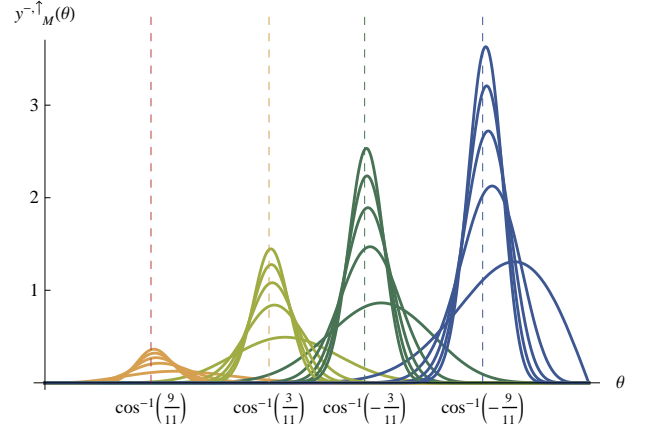


FIG. 2: Conditional probability distributions for the qubit to be in the  $|\uparrow\rangle$  state when the total system is in its ground state. Values of  $S$  and  $M/\tilde{S}$ , as well as dashed lines meaning, as in Fig. 1.

probability distribution for the qubit to be, say, in the state  $|\uparrow\rangle$ , reads

$$y_M^{\pm,\uparrow}(\theta) = p_M^\pm(\theta) \cos^2 \frac{\Theta_M^\pm(\theta)}{2}; \quad (21)$$

In the case of the GS multiplet, despite  $|\phi_M^-(\Omega)\rangle$  being independent on  $M$ , due to  $\Theta_M^- = \pi - \theta$  for all  $M$ , the conditional probability distribution (21) inherits the dependence on  $M$  from the environment, as clearly seen in Fig.2. Notice, that such dependence accounts for the antiferromagnetic character of the interaction term (9) by representing the counteralignment of the qubit with respect to the total environmental spin.

Let us now further comment on the parametrized qubit states (19): it is easily shown that one can always define magnetic fields such that the kets  $|\phi_M^\pm(\Omega)\rangle$  are the ground (“−”) and excited (“+”) states of the corresponding Zeeman terms. In fact, defining  $\mathbf{h}_M^\pm(\Omega) \equiv \varepsilon^\pm \mathbf{n}(\Theta_M^\pm, \varphi)$ , with  $\mathbf{n}(\Theta_M^\pm, \varphi) \equiv (\sin \Theta_M^\pm \cos \varphi, \sin \Theta_M^\pm \sin \varphi, \cos \Theta_M^\pm)$  and  $\Theta_M^\pm$  from Eq.(20), it is easily verified that  $\mathbf{h}_M^\pm \cdot$

$\frac{\sigma}{2}|\phi_M^\pm(\Omega)\rangle = \frac{\varepsilon^\pm}{2}|\phi_M^\pm(\Omega)\rangle$ , for all  $M$ . By further requiring

$$\int d\mu(\Omega)|\chi(\Omega)|^2\langle\phi_M^\pm(\Omega)|\mathbf{h}_M^\pm\cdot\frac{\boldsymbol{\sigma}}{2}|\phi_M^\pm(\Omega)\rangle = \langle\Psi_M^\pm|H_{AB}|\Psi_M^\pm\rangle, \quad (22)$$

one can also fix the field intensity,  $\varepsilon^\pm = \pm\frac{g}{N}(\tilde{S}\mp 1)$ . The above result means that when the star is in one of its eigenstates  $|\Psi_M^\pm\rangle$  the central spin can be described by a continuous collection of pure states which are the eigenstates of  $\mathbf{h}_M^\pm(\Omega)\cdot\frac{\boldsymbol{\sigma}}{2}$ , with the direction of the fields distributed according to the environmental quantum probability  $|\chi_M^\pm(\Omega)|^2$ . This also provides a key to the reading of Fig.1, whose lower panel visualizes the qubit response to the application of a field with direction distributed as shown in the upper panel. Picturing the qubit in terms of the above "effective" magnetic fields, one is naturally led to the last part of our work, namely the crossover from a quantum to a classical environment.

#### IV. THE LARGE- $S$ LIMIT.

From a direct calculation one can see that

$$p_M^\pm(\theta) \xrightarrow{S\rightarrow\infty} \delta(\theta - \vartheta_M), \quad (23)$$

meaning that the variable describing the polar angle of the environmental coherent states,  $\theta$ , gets frozen to the value  $\vartheta_M$ , which is fixed by the state of the global system,  $|\Psi_M^\pm\rangle$ . The qubit parametrized states consequently collapse into the kets (19) with  $\Theta_M^+ = \vartheta_M$  and  $\Theta_M^- = \pi - \vartheta_M$ , hereafter indicated by  $|\phi_M^\pm(\varphi)\rangle$ . Notice that the dependence on  $\theta$  is fully removed by the above classical limit, but all the azimuthal angles  $\varphi \in [0, 2\pi)$  are still allowed.

The kets  $|\phi_M^\pm(\varphi)\rangle$  are the stationary states of a qubit in a field  $\frac{g}{2}\mathbf{n}_M^\pm(\varphi)$ , with  $\mathbf{n}_M^\pm(\varphi) \equiv \mathbf{n}(\Theta_M^\pm, \varphi)$  and  $\Theta_M^\pm$  as above (notice that  $S \rightarrow \infty$  implies  $S/N \rightarrow 1/2$ , so that  $\varepsilon^\pm$  converge to  $g/2$ ). From this perspective, the model after Eq.(23) is that of a qubit in a pure state, which is one of the eigenstates of a local parametric Hamiltonian  $\frac{g}{2}\mathbf{n}_M^\pm(\varphi)\cdot\frac{\boldsymbol{\sigma}}{2}$ , where the only parametric dependence left is that on the azimuthal angle of the field. Put it this way, with the qubit in a pure state, one seems to loose the connection with the original composite quantum system and, in particular, with the fact that the parametric dependence is indeed a consequence of the entangled structure of its global state  $|\Psi_M^\pm\rangle$ , as seen in the second Section.

Let us hence take a different point of view: Leaving aside the local parametric Hamiltonian, we focus on the qubit state. Once  $M$  is fixed, the qubit is described by all the kets  $\{|\phi_M^\pm(\varphi)\rangle\}_{\varphi\in[0,2\pi)}$ , which define a mixed state. In fact, as these states are all equally likely, the corresponding density matrix is given by the ensemble average of their projectors, i.e.  $\rho_{\sigma^\pm}^{M} = \frac{1}{2\pi} \int d\varphi |\phi_M^\pm(\varphi)\rangle\langle\phi_M^\pm(\varphi)|$ . Due to the  $\varphi$ -integral, that makes the off-diagonal elements vanish,  $\rho_{\sigma^\pm}^{M}$  coincides with the reduced density

matrix of the qubit in the original fully quantum model, i.e.  $\text{Tr}_{\text{ring}}|\Psi_M^\pm\rangle\langle\Psi_M^\pm|$ . Therefore, not only the qubit can still be characterized by a finite Von Neumann entropy,  $E^{\text{VN}}(\rho_\sigma)$ , but this turns out to be equal to the entanglement between the qubit and its environment before the quantum limit of the latter is taken, i.e.

$$\mathcal{E}_{\sigma S}(|\Psi_M^\pm\rangle) = E^{\text{VN}}(\rho_{\sigma^\pm}^M). \quad (24)$$

The above statistical picture fits in the scheme proposed in Ref.<sup>45</sup> for evaluating the Von Neumann entropy of a qubit in whatever mixed state  $\rho_\sigma$ . It is there shown that when a qubit is described by an ensemble of  $K$  equally likely pure states, corresponding to the Bloch vectors  $\{\mathbf{n}_p\}_{p\in P}$ , with  $P$  some parameter space, its density matrix is  $\rho_\sigma = \frac{1}{2}(1 + \bar{\mathbf{n}}\cdot\boldsymbol{\sigma})$ , with the "average" Bloch vector  $\bar{\mathbf{n}}$  defined as  $\bar{\mathbf{n}} \equiv \frac{1}{K} \sum_p \mathbf{n}_p$ , and its Von Neumann entropy can be written as

$$E^{\text{VN}}(\rho_\sigma) = -h\left[\frac{1}{2}(1 + |\bar{\mathbf{n}}|)\right] \quad (25)$$

Furthermore, given the ensemble  $\{\mathbf{n}_p\}_{p\in P}$ , one can choose a sequence  $\mathbf{n}_{p_1} \rightarrow \mathbf{n}_{p_2} \rightarrow \mathbf{n}_{p_3} \dots \rightarrow \mathbf{n}_{p_K}$  thus defining a curve in the parameter space  $P$ . To this curve corresponds a Pancharatnam phase<sup>46,47</sup>  $\gamma \equiv \arg\{\text{Tr}[\Pi_{i=1}^{K-1}\langle\mathbf{n}_{p_i}|\mathbf{n}_{p_{i+1}}\rangle\langle\mathbf{n}_{p_K}|]\}$ , for which several relations with geometrical properties of the ensemble  $\{\mathbf{n}_p\}_{p\in P}$  can be found<sup>45</sup>. Interpreting the sum defining  $\bar{\mathbf{n}}$  as an ensemble average, and generalizing the above construction to the continuum case, we find that the mixed state in which the central qubit of the spin star is left, after the classical limit of the ring is taken, corresponds to

$$\bar{\mathbf{n}}_M^\pm \equiv \int d\varphi \mathbf{n}_M^\pm(\varphi) = (0, 0, \pm \cos \vartheta_M) \quad (26)$$

so that  $E^{\text{VN}}(\rho_{\sigma^\pm}^M) = -h\left[\frac{1}{2}(1 + \cos \vartheta_M)\right]$ , consistently with Eqs.(24) and (14). Furthermore, given our continuous ensemble, we can choose the closed sequence corresponding to the clock(-) or counterclock(+) wise oriented (with respect to the qubit quantization axis) line of latitude  $\vartheta_M$ , for which the Pancharatnam phase can be shown to be

$$\gamma_\pm = \pi(1 \pm |\bar{\mathbf{n}}_M^\pm|) = \pi(1 \pm \cos \vartheta_M), \quad (27)$$

which finally allows us to establish, via Eqs.(24) to (27), the relation

$$\mathcal{E}_{\sigma S} = -h\left[\frac{\gamma_\pm}{2\pi}\right]. \quad (28)$$

Notice that the phase  $\gamma^\pm$  as obtained via the above construction is exactly the Berry phase which is picked by the eigenstates of a qubit in a magnetic field  $\frac{g}{2}\mathbf{n}_M^\pm(\varphi)$  adiabatically rotating around the quantization axis with fixed polar angle  $\vartheta_M$ , a problem that pertains to the study of models with parametric Hamiltonians. However, in the usual approach to the study of these models

the parametric dependence is not derived as the effect of some quantum environment, but rather assumed a priori, a viewpoint that leaves no space for entanglement properties, hampering the disclosure of their possible relation with geometrical phases.

We underline that the relation Eq.(28) specifically holds just for the particular model here considered. However, we believe that the reasoning behind its derivation has a more general content, that pertains to the analysis of the relation between geometrical properties of quantum systems and the structure, or dynamical evolution, of their states<sup>35,36,48–55</sup>. In particular we underline that the entanglement of the state of the original composite quantum system is the ultimate responsible for the dependence on the coherent state variables to be conveyed from the environment to either the states or the local parametric Hamiltonian (depending on what point of view is adopted) of the principal system. In the limit of a classical environment, these variables become the parameters by exploring whose space the principal system can experience geometrical effects.

## V. CONCLUSIONS

In this work we have proposed a method for studying the behaviour of an open quantum system along the quantum-to-classical crossover of its environment. The method, which is based on an exact, parametric representation for whatever state of an isolated system, is an original tool for dealing with phenomena which manifest themselves, and can be interpreted, very differently depending on the way the environment is modelled, not only in physical (see for instance Refs.<sup>56–58</sup>) but also in chemical and biological processes<sup>59–61</sup>. As a first direct outcome, this work clarifies why modelling a quantum system with a parametric Hamiltonian implies the existence of an environment ("the rest of the Universe" to use Berry's words<sup>34</sup>) and shows that a non-trivial parametric dependence can arise if and only if such environment is entangled with the system itself. One of the most relevant consequences of the above statement is that the emergence of observable (i.e. gauge-invariant) quantities which are not eigenvalues of Hermitian operators of the system under analysis, such as the Berry's phase, turns out to be related not only to the fact that an environment exists<sup>47</sup>, but specifically to the condition that the system be entangled with its environment. In fact, in the specific case of the spin-star here considered, Eq.(28) establishes that the entanglement between the qubit and its environment can be determined by measuring the observable Berry's phase characterizing the related model of a closed

qubit in a magnetic field, which suggests a possible way for experimentally access entanglement properties via the observation of gauge-invariant phases.

We conclude by mentioning some possible developments of this work that we think are worth pursuing. First of all, one might exploit the peculiar properties characterizing the dynamics of generalized coherent states, in order to relate a possible dynamical evolution of the global system to that of the principal one. Indeed, the formal scheme here presented opens the possibility of using established approaches for dealing with quantum dynamics in phase space, such as the the path-integral formalism, the adiabatic perturbation theory, the Born-Oppenheimer approximation, and generalizations to curved phase spaces of multi-configurational Ehrefest methods, as tools for taking into account the effects of the environment on the principal system and vice versa<sup>19,62–66</sup>. Moreover, the approach here presented can be equivalently implemented in terms of generalized coherent states of any type<sup>37</sup>, so that different physical systems can be taken into consideration, such as those belonging to the spin-bosons family. As for the spin- $\frac{1}{2}$  star with frustration, it is worth mentioning that different types of interaction between the environmental spins, in particular the antiferromagnetic Lieb-Mattis and Heisenberg-on-a-square-lattice ones, define exactly solvable models<sup>67</sup> that can be treated in the very same framework here proposed. This expands the set of real physical systems where to look for a possible experimental analysis of our results. Finally, we find particularly intriguing the idea of effectively studying the spin- $\frac{1}{2}$  star with frustration by quantum simulators (see e.g. Ref.<sup>68</sup> and references therein, and Refs.<sup>69</sup>): In fact, the possibility of tuning the interaction parameters, which is recognized as one of the key features of quantum simulators, might allow the variation of the value of  $S$ , by acting on the frustration ratio  $k/g$ , thus giving access to an experimental analysis of the crossover from a quantum to a classical environment.

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